

An Extension of Jensen's Inequality on Time Scales

Fu-Hsiang Wong

*Department of Mathematics, National Taipei Education University,
134, Ho-Ping E. Rd, Sec2, Taipei 10659,
Taiwan, Republic of China
E-mail: wong@tea.ntue.edu.tw*

Cheh-Chih Yeh

*Department of Information Management,
Lunghwa University of Science and Technology,
Kueishan Taoyuan, 33306 Taiwan, Republic of China
E-mail: cycyeh@mail.lhu.edu.tw or chehchihyeh@yahoo.com.tw*

Wei-Cheng Lian

*Department of Information Management,
National Kaohsiung Marine University,
No.142, Hai Chuan Road, Nan-Tzu Dist, Kaohsiung,
Taiwan, Republic of China
E-mail: wclian@mail.nkimt.edu.tw*

Abstract

The renowned Jensen inequality is established on time scales as follows:

$$f \left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s} \right) \leq \frac{\int_a^b |h(s)|f(g(s))\Delta s}{\int_a^b |h(s)|\Delta s},$$

if f , g and h satisfy some suitable conditions.

AMS subject classification: 26B25, 26D15.

Keywords: Time scales, Jensen's inequality, convex, delta derivative.

1. Introduction

The Jensen inequality [9] is of great interest in differential and difference equations, and other areas of mathematics. The original Jensen inequality is as follows:

If $g \in C([a, b], (c, d))$ and $f \in C((c, d), \mathbb{R})$ are convex, then

$$f\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{\int_a^b f(g(s))ds}{b-a}.$$

Many authors have dealt with this renowned inequality, see, for example, Agarwal et al. [1] and the references therein. The Jensen inequality has been extended to time scales by Agarwal, Bohner, and Peterson as follows (see [1, 3]):

Theorem 1.1. If $g \in C_{rd}([a, b], (c, d))$ and $f \in C((c, d), \mathbb{R})$ are convex, then

$$f\left(\frac{\int_a^b g(s)\Delta s}{b-a}\right) \leq \frac{\int_a^b f(g(s))\Delta s}{b-a}.$$

The purpose of this paper is to generalize Theorem 1.1 to a more general case. For related results, we refer to [4, 8, 9].

Now, we briefly introduce the time scales calculus and refer to Aulbach and Hilger [2] and Hilger [6] and the books [3, 7] for further details.

By a time scale \mathbb{T} we mean any closed subset of \mathbb{R} with order and topological structure present in a canonical way. Since a time scale \mathbb{T} may or may not be connected, we need the concept of jump operators.

Definition 1.2. Let $t \in \mathbb{T}$, where \mathbb{T} is a time scale. The two mappings

$$\sigma, \rho : \mathbb{T} \rightarrow \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} | s < t\}$$

are called the jump operators. If $\sigma(t) > t$, t is right-scattered. If $\rho(t) < t$, t is left-scattered. If $\sigma(t) = t$, t is right-dense. If $\rho(t) = t$, t is left-dense.

Definition 1.3. A mapping $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous if it satisfies the following two conditions:

- (A) f is continuous at each right-dense point or maximal element of \mathbb{T} ,
- (B) the left-sided limit $\lim_{s \rightarrow t^-} f(s) = f(t^-)$ exists at each left-dense point t of \mathbb{T} .

Throughout this paper, we suppose that

- (a) $\mathbb{R} = (-\infty, +\infty)$;

- (b) \mathbb{T} is a time scale;
- (c) an interval means the intersection of a real interval with the given time scale;
- (d)

$$C_{rd}(\mathbb{T}, \mathbb{R}) := \{f \mid f : \mathbb{T} \rightarrow \mathbb{R} \text{ is an rd-continuous function}\};$$

- (e)

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Definition 1.4. If $f : \mathbb{T} \rightarrow \mathbb{R}$, then $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$.

Definition 1.5. Assume $x : \mathbb{T} \rightarrow \mathbb{R}$ and fix $t \in \mathbb{T}^\kappa$. We define $x^\Delta(t)$ as the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. Here $x^\Delta(t)$ is said to be the **delta derivative** of x at t .

It can be shown that if $x : \mathbb{T} \rightarrow \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition 1.6. A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case, we define the integral of f by

$$\int_s^t f(\tau) \Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}$.

2. Main Result

To establish our main result, we need the following lemma which is [5, Exercise 3.42C].

Lemma 2.1. Let $f \in C((c, d), \mathbb{R})$ be convex. Then, for each $t \in (c, d)$, there exists $a_t \in \mathbb{R}$ such that

$$f(x) - f(t) \geq a_t(x - t) \quad \text{for all } x \in (c, d).$$

If f is strictly convex, then the inequality sign “ \geq ” in the above inequality should be replaced by “ $>$ ”.

We are in a position to state and prove our main result.

Theorem 2.2. (Jensen's inequality on time scales) Let $g \in C_{rd}([a, b], (c, d))$ and $h \in C_{rd}([a, b], \mathbb{R})$ with

$$\int_a^b |h(s)| \Delta s > 0,$$

where $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $f \in C((c, d), \mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \leq \frac{\int_a^b |h(s)|f(g(s))\Delta s}{\int_a^b |h(s)|\Delta s}.$$

If f is strictly convex, then the inequality sign " \leq " in the above inequality should be replaced by " $<$ ".

Proof. Since f is convex, it follows from Lemma 2.1 that for each $t \in (c, d)$, there exists $a_t \in \mathbb{R}$ such that

$$f(x) - f(t) \geq a_t(x - t)$$

for all $x \in (c, d)$. Let

$$t = \frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}.$$

Thus,

$$\begin{aligned} & \int_a^b |h(s)|f(g(s))\Delta s - \left(\int_a^b |h(s)|\Delta s\right) f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \\ &= \int_a^b |h(s)|f(g(s))\Delta s - \left(\int_a^b |h(s)|\Delta s\right) f(t) \\ &= \int_a^b |h(s)|\{f(g(s)) - f(t)\}\Delta s \\ &\geq a_t \int_a^b |h(s)|\{g(s) - t\}\Delta s \\ &= a_t \left\{ \int_a^b |h(s)|g(s)\Delta s - t \int_a^b |h(s)|\Delta s \right\} \\ &= a_t \left\{ \int_a^b |h(s)|g(s)\Delta s - \int_a^b |h(s)|g(s)\Delta s \right\} \\ &= 0, \end{aligned}$$

which completes our proof. ■

Letting $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ in Theorem 2.2, we have the following two corollaries which improve [8, Theorems 2 and 3 on p. 109], respectively.

Corollary 2.3. ($\mathbb{T} = \mathbb{R}$) Let $g, h : [a, b] \rightarrow \mathbb{R}$ be integrable with $\int_a^b |h(x)|dx > 0$. If $f \in C((c, d), \mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b |h(x)|g(x)dx}{\int_a^b |h(x)|dx}\right) \leq \frac{\int_a^b |h(x)|f(g(x))dx}{\int_a^b |h(x)|dx},$$

where $g([a, b]) \subseteq (c, d)$.

Corollary 2.4. ($\mathbb{T} = \mathbb{Z}$) Let f be a convex function. Then for any x_1, x_2, \dots, x_n and $c_1, c_2, \dots, c_n \in \mathbb{Z}$ with $\sum_{k=1}^n c_k > 0$,

$$f\left(\frac{\sum_{k=1}^n c_k x_k}{\sum_{k=1}^n c_k}\right) \leq \frac{\sum_{k=1}^n c_k f(x_k)}{\sum_{k=1}^n c_k}.$$

Remark 2.5. If the condition “ f is convex” is changed into “ f is concave”, then the inequality signs of the conclusions in the above theorems and corollaries should be replaced by “ \geq ”.

Remark 2.6. Let $g(t) \geq 0$ on $[a, b]$ and $f(t) = t^\alpha$ on $[0, +\infty)$ in Theorem 2.2. It is clear that f is convex on $[0, +\infty)$ for $\alpha < 0$ or $\alpha > 1$, and f is concave on $[0, +\infty)$ for $\alpha \in (0, 1)$. Therefore,

$$\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right)^\alpha \leq \frac{\int_a^b |h(s)|g^\alpha(s)\Delta s}{\int_a^b |h(s)|\Delta s}, \quad \text{if } \alpha < 0 \text{ or } \alpha > 1;$$

$$\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right)^\alpha \geq \frac{\int_a^b |h(s)|g^\alpha(s)\Delta s}{\int_a^b |h(s)|\Delta s}, \quad \text{if } \alpha \in (0, 1).$$

Remark 2.7. Let $g(t) > 0$ on $[a, b]$ and $f(t) = \ln(t)$ on $(0, +\infty)$ in Theorem 2.2. It is clear that f is concave on $(0, +\infty)$. Therefore,

$$\ln\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \geq \frac{\int_a^b |h(s)|\ln(g(s))\Delta s}{\int_a^b |h(s)|\Delta s}.$$

3. Applications

Applying Jensen's inequality (Theorem 2.2), we have the following three theorems.

Theorem 3.1. Let $p, h \in C_{rd}([a, b], [0, \infty))$ with $\int_a^b p(s)h(s)\Delta s > 0$ and $\int_a^b \frac{p(s)}{h(s)}\Delta s > 0$. Then

$$\frac{\int_a^b \frac{p(s)}{h(s)} \ln(h(s))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \frac{\int_a^b p(s)h(s) \ln(h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}.$$

Proof. Since $f(x) = -\ln(x)$ is strictly convex, it follows from the Jensen inequality (Theorem 2.2) that

$$f\left(\frac{\int_a^b p(s)\frac{1}{h(s)}\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{\int_a^b p(s)f\left(\frac{1}{h(s)}\right)\Delta s}{\int_a^b p(s)\Delta s}.$$

That is,

$$-\ln\left(\frac{\int_a^b \frac{p(s)}{h(s)}\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{-\int_a^b p(s)\ln\left(\frac{1}{h(s)}\right)\Delta s}{\int_a^b p(s)\Delta s},$$

which implies

$$\ln\left(\frac{\int_a^b p(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}\right) < \frac{\int_a^b p(s)\ln(h(s))\Delta s}{\int_a^b p(s)\Delta s}.$$

Thus,

$$\frac{\int_a^b p(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \exp\left(\frac{\int_a^b p(s)\ln(h(s))\Delta s}{\int_a^b p(s)\Delta s}\right).$$

Similarly,

$$\frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b p(s)\Delta s} = \frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b \frac{p(s)h(s)}{h(s)}\Delta s} < \exp\left(\frac{\int_a^b p(s)h(s)\ln(h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}\right).$$

It follows from this and Jensen's inequality with respect to the strictly convex function \exp that

$$\begin{aligned} \exp\left(\frac{\int_a^b \frac{p(s)}{h(s)} \ln(h(s))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}\right) &< \frac{\int_a^b \frac{p(s)}{h(s)} \exp(\ln(h(s)))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} \\ &= \frac{\int_a^b \frac{p(s)}{h(s)} h(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\int_a^b p(s)\Delta t}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \exp\left(\frac{\int_a^b p(s)\ln(h(s))\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{\int_a^b p(s)\exp(\ln(h(s)))\Delta s}{\int_a^b p(s)\Delta s} \\
 &= \frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b p(s)\Delta s} < \exp\left(\frac{\int_a^b p(s)h(s)\ln(h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}\right),
 \end{aligned}$$

which completes the proof. ■

Theorem 3.2. (Hölder's inequality) Let $h, f, g \in C_{rd}([a, b], [0, \infty))$ with

$$\int_a^b h(x)g^q(x)\Delta x > 0.$$

If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then

$$\int_a^b h(x)f(x)g(x)\Delta x \leq \left(\int_a^b h(x)f^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b h(x)g^q(x)\Delta x\right)^{\frac{1}{q}}.$$

Proof. Taking $f(x) = x^p$ and letting $g, |h(x)|$ be replaced by $fg^{-\frac{q}{p}}, hg^q$ in Theorem 2.2, respectively, we obtain

$$\left(\frac{\int_a^b h(x)g^q(x)f(x)g^{-\frac{q}{p}}(x)\Delta x}{\int_a^b h(x)g^q(x)\Delta x}\right)^p \leq \frac{\int_a^b h(x)g^q(x)(f(x)g^{-\frac{q}{p}}(x))^p\Delta x}{\int_a^b h(x)g^q(x)\Delta x}.$$

This and $\frac{1}{p} + \frac{1}{q} = 1$ imply

$$\int_a^b h(x)f(x)g(x)\Delta x \leq \left(\int_a^b h(x)f^p(x)\Delta x\right)^{\frac{1}{p}} \left(\int_a^b h(x)g^q(x)\Delta x\right)^{\frac{1}{q}}.$$

Theorem 3.3. Let $h, f, g \in C_{rd}([a, b], [0, \infty))$. Then

(a)

$$\left[\left(\int_a^b hf\Delta x\right)^r + \left(\int_a^b hg\Delta x\right)^r\right]^{\frac{1}{r}} \leq \int_a^b h(f^r + g^r)^{\frac{1}{r}}\Delta x, \text{ if } r > 1;$$

(b)

$$\left[\left(\int_a^b hf\Delta x\right)^r + \left(\int_a^b hg\Delta x\right)^r\right]^{\frac{1}{r}} \geq \int_a^b h(f^r + g^r)^{\frac{1}{r}}\Delta x, \text{ if } 0 < r < 1.$$

Proof. (a) Clearly, $\varphi(x) = (1 + x^r)^{\frac{1}{r}}$ is convex on $(0, \infty)$. Hence, by Theorem 2.2,

$$\left[1 + \left(\int_a^b h(x)f(x)\Delta x \right)^r \right]^{\frac{1}{r}} \leq \int_a^b h(x)(1 + f^r(x))^{\frac{1}{r}} \Delta x .$$

Letting h and f be replaced by $\frac{hf}{\int_a^b hf\Delta x}$ and $\frac{g}{f}$ in the above inequality, respectively, we get our desired result.

Similarly, we can prove case (b). ■

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