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An Extension of Jensen's Inequality on Time Scales

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Abstract

The renowned Jensen inequality is established on time scales as follows:

$$f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \leq \frac{\int_a^b |h(s)|f(g(s))\Delta s}{\int_a^b |h(s)|\Delta s},$$

if f, g and h satisfy some suitable conditions.

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1. Introduction

The Jensen inequality [9] is of great interest in differential and difference equations, and other areas of mathematics. The original Jensen inequality is as follows:

If $g \in C([a,b],(c,d))$ and $f \in C((c,d),\mathbb{R})$ are convex, then

$$f\left(\frac{\int_a^b g(s)ds}{b-a}\right) \le \frac{\int_a^b f(g(s))ds}{b-a}.$$

Many authors have dealt with this renowned inequality, see, for example, Agarwal et al. [1] and the references therein. The Jensen inequality has been extended to time scales by Agarwal, Bohner, and Peterson as follows (see [1,3]):

Theorem 1.1. If $g \in C_{rd}([a,b],(c,d))$ and $f \in C((c,d),\mathbb{R})$ are convex, then

$$f\left(\frac{\int_a^b g(s)\Delta s}{b-a}\right) \le \frac{\int_a^b f(g(s))\Delta s}{b-a}.$$

The purpose of this paper is to generalize Theorem 1.1 to a more general case. For related results, we refer to [4,8,9].

Now, we briefly introduce the time scales calculus and refer to Aulbach and Hilger [2] and Hilger [6] and the books [3,7] for further details.

By a time scale $\mathbb{T}$ we mean any closed subset of $\mathbb{R}$ with order and topological structure present in a canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

Definition 1.2. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale. The two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} | s < t\}$$

are called the jump operators. If $\sigma(t) > t$, t is right-scattered. If $\rho(t) < t$, t is left-scattered. If $\sigma(t) = t$, t is right-dense. If $\rho(t) = t$, t is left-dense.

Definition 1.3. A mapping $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it satisfies the following two conditions:

(A) f is continuous at each right-dense point or maximal element of $\mathbb{T}$,

(B) the left-sided limit $\lim_{s \to t^-} f(s) = f(t^-)$ exists at each left-dense point t of $\mathbb{T}$.

Throughout this paper, we suppose that

(a) $\mathbb{R} = (-\infty, +\infty)$;

(b) $\mathbb{T}$ is a time scale;

(c) an interval means the intersection of a real interval with the given time scale;

(d)
$$C_{rd}(\mathbb{T},\mathbb{R}) := \{f \mid f:\mathbb{T}\to\mathbb{R} \text{ is an rd-continuous function}\};$$

(e)
$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}-\{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Definition 1.4. If $f:\mathbb{T}\to\mathbb{R}$, then $f^\sigma:\mathbb{T}\to\mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))$$

for all $t\in\mathbb{T}$.

Definition 1.5. Assume $x:\mathbb{T}\to\mathbb{R}$ and fix $t\in\mathbb{T}^\kappa$. We define $x^\Delta(t)$ as the number (provided it exists) with the property that given any $\epsilon>0$, there is a neighborhood U of t such that

$$|[x(\sigma(t))-x(s)]-x^\Delta(t)[\sigma(t)-s]| < \epsilon|\sigma(t)-s|,$$

for all $s\in U$. Here $x^\Delta(t)$ is said to be the **delta derivative** of x at t.

It can be shown that if $x:\mathbb{T}\to\mathbb{R}$ is continuous at $t\in\mathbb{T}$ and t is right-scattered, then

$$x^\Delta(t) = \frac{x(\sigma(t))-x(t)}{\sigma(t)-t}.$$

Definition 1.6. A function $F:\mathbb{T}\to\mathbb{R}$ is an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t\in\mathbb{T}^\kappa$. In this case, we define the integral of f by

$$\int_s^t f(\tau)\ \Delta\tau = F(t)-F(s)$$

for $s,t\in\mathbb{T}$.

2. Main Result

To establish our main result, we need the following lemma which is [5, Exercise 3.42C].

Lemma 2.1. Let $f\in C((c,d),\mathbb{R})$ be convex. Then, for each $t\in(c,d)$, there exists $a_t\in\mathbb{R}$ such that

$$f(x)-f(t)\geq a_t(x-t) \quad \text{for all} \quad x\in(c,d).$$

If f is strictly convex, then the inequality sign "$\geq$" in the above inequality should be replaced by "$>$".

We are in a position to state and prove our main result.

Theorem 2.2. (Jensen's inequality on time scales) Let $g \in C_{rd}([a,b],(c,d))$ and $h \in C_{rd}([a,b],\mathbb{R})$ with

$$\int_a^b |h(s)|\Delta s > 0,$$

where $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. If $f \in C((c,d),\mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \le \frac{\int_a^b |h(s)|f(g(s))\Delta s}{\int_a^b |h(s)|\Delta s}.$$

If f is strictly convex, then the inequality sign "$\le$" in the above inequality should be replaced by "$<$".

Proof. Since f is convex, it follows from Lemma 2.1 that for each $t \in (c,d)$, there exists $a_t \in \mathbb{R}$ such that

$$f(x) - f(t) \ge a_t(x-t)$$

for all $x \in (c,d)$. Let

$$t = \frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}.$$

Thus,

$$\begin{aligned}
&\int_a^b |h(s)|f(g(s))\Delta s - \left(\int_a^b |h(s)|\Delta s\right) f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \\
&= \int_a^b |h(s)|f(g(s))\Delta s - \left(\int_a^b |h(s)|\Delta s\right) f(t) \\
&= \int_a^b |h(s)|\{f(g(s)) - f(t)\}\Delta s \\
&\ge a_t \int_a^b |h(s)|\{g(s) - t\}\Delta s \\
&= a_t \left\{\int_a^b |h(s)|g(s)\Delta s - t\int_a^b |h(s)|\Delta s\right\} \\
&= a_t \left\{\int_a^b |h(s)|g(s)\Delta s - \int_a^b |h(s)|g(s)\Delta s\right\} \\
&= 0,
\end{aligned}$$

which completes our proof. ■

Letting $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ in Theorem 2.2, we have the following two corollaries which improve [8, Theorems 2 and 3 on p. 109], respectively.

Corollary 2.3. **($\mathbb{T} = \mathbb{R}$)** Let $g, h : [a, b] \to \mathbb{R}$ be integrable with $\int_a^b |h(x)|dx > 0$. If $f \in C((c, d), \mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b |h(x)|g(x)dx}{\int_a^b |h(x)|dx}\right) \leq \frac{\int_a^b |h(x)|f(g(x))dx}{\int_a^b |h(x)|dx},$$

where $g([a, b]) \subseteq (c, d)$.

Corollary 2.4. **($\mathbb{T} = \mathbb{Z}$)** Let f be a convex function. Then for any $x_1, x_2, \ldots, x_n$ and $c_1, c_2, \ldots, c_n \in \mathbb{Z}$ with $\sum\limits_{k=1}^{n} c_k > 0$,

$$f\left(\frac{\sum\limits_{k=1}^{n} c_k x_k}{\sum\limits_{k=1}^{n} c_k}\right) \leq \frac{\sum\limits_{k=1}^{n} c_k f(x_k)}{\sum\limits_{k=1}^{n} c_k}.$$

Remark 2.5. If the condition "f is convex" is changed into "f is concave", then the inequality signs of the conclusions in the above theorems and corollaries should be replaced by "$\geq$".

Remark 2.6. Let $g(t) \geq 0$ on $[a, b]$ and $f(t) = t^\alpha$ on $[0, +\infty)$ in Theorem 2.2. It is clear that f is convex on $[0, +\infty)$ for $\alpha < 0$ or $\alpha > 1$, and f is concave on $[0, +\infty)$ for $\alpha \in (0, 1)$. Therefore,

$$\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right)^\alpha \leq \frac{\int_a^b |h(s)|g^\alpha(s)\Delta s}{\int_a^b |h(s)|\Delta s}, \qquad \text{if} \quad \alpha < 0 \quad \text{or} \quad \alpha > 1;$$

$$\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right)^\alpha \geq \frac{\int_a^b |h(s)|g^\alpha(s)\Delta s}{\int_a^b |h(s)|\Delta s}, \qquad \text{if} \quad \alpha \in (0, 1).$$

Remark 2.7. Let $g(t) > 0$ on $[a, b]$ and $f(t) = \ln(t)$ on $(0, +\infty)$ in Theorem 2.2. It is clear that f is concave on $(0, +\infty)$. Therefore,

$$\ln\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \geq \frac{\int_a^b |h(s)|\,\ln(g(s))\Delta s}{\int_a^b |h(s)|\Delta s}.$$

3. Applications

Applying Jensen's inequality (Theorem 2.2), we have the following three theorems.

Theorem 3.1. Let $p, h \in C_{rd}([a,b],[0,\infty))$ with $\int_a^b p(s)h(s)\Delta s > 0$ and $\int_a^b \frac{p(s)}{h(s)}\Delta s > 0$. Then

$$\frac{\int_a^b \frac{p(s)}{h(s)} \ \ln(h(s))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \frac{\int_a^b p(s)h(s)\ln(h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}.$$

Proof. Since $f(x) = -\ln(x)$ is strictly convex, it follows from the Jensen inequality (Theorem 2.2) that

$$f\left(\frac{\int_a^b p(s)\frac{1}{h(s)}\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{\int_a^b p(s) f(\frac{1}{h(s)})\Delta s}{\int_a^b p(s)\Delta s}.$$

That is,

$$-\ln\left(\frac{\int_a^b \frac{p(s)}{h(s)}\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{-\int_a^b p(s) \ \ln(\frac{1}{h(s)})\Delta s}{\int_a^b p(s)\Delta s},$$

which implies

$$\ln\left(\frac{\int_a^b p(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}\right) < \frac{\int_a^b p(s) \ \ln \ (h(s))\Delta s}{\int_a^b p(s)\Delta s}.$$

Thus,

$$\frac{\int_a^b p(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \ \exp \ \left(\frac{\int_a^b p(s) \ \ln \ (h(s))\Delta s}{\int_a^b p(s)\Delta s}\right).$$

Similarly,

$$\frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b p(s)\Delta s} = \frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b \frac{p(s)h(s)}{h(s)}\Delta s} < \exp \ \left(\frac{\int_a^b p(s)h(s) \ \ln \ (h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}\right).$$

It follows from this and Jensen's inequality with respect to the strictly convex function $\exp$ that

$$\exp \ \left(\frac{\int_a^b \frac{p(s)}{h(s)} \ \ln \ (h(s))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}\right) < \frac{\int_a^b \frac{p(s)}{h(s)} \ \exp \ (\ln \ (h(s)))\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}$$

$$= \frac{\int_a^b \frac{p(s)}{h(s)} h(s)\Delta s}{\int_a^b \frac{p(s)}{h(s)}\Delta s}$$

$$= \frac{\int_a^b p(s)\Delta t}{\int_a^b \frac{p(s)}{h(s)}\Delta s} < \exp\left(\frac{\int_a^b p(s)\ \ln\ (h(s))\Delta s}{\int_a^b p(s)\Delta s}\right) < \frac{\int_a^b p(s)\exp\ (\ln\ (h(s)))\Delta s}{\int_a^b p(s)\Delta s}$$

$$= \frac{\int_a^b p(s)h(s)\Delta s}{\int_a^b p(s)\Delta s} < \exp\left(\frac{\int_a^b p(s)h(s)\ \ln\ (h(s))\Delta s}{\int_a^b p(s)h(s)\Delta s}\right),$$

which completes the proof. ■

Theorem 3.2. (Hölder's inequality) Let $h,\ f,\ g \in C_{rd}([a,b],[0,\infty))$ with

$$\int_a^b h(x)g^q(x)\Delta x > 0.$$

If $\dfrac{1}{p}+\dfrac{1}{q}=1$ with $p>1$, then

$$\int_a^b h(x)f(x)g(x)\Delta x \le \left(\int_a^b h(x)f^p(x)\Delta x\right)^{\frac{1}{p}}\left(\int_a^b h(x)g^q(x)\Delta x\right)^{\frac{1}{q}}.$$

Proof. Taking $f(x) = x^p$ and letting g, $|h(x)|$ be replaced by $fg^{-\frac{q}{p}}$, hg^q in Theorem 2.2, respectively, we obtain

$$\left(\frac{\displaystyle\int_a^b h(x)g^q(x)f(x)g^{-\frac{q}{p}}(x)\Delta x}{\displaystyle\int_a^b h(x)g^q(x)\Delta x}\right)^p \le \frac{\displaystyle\int_a^b h(x)g^q(x)\big(f(x)g^{-\frac{q}{p}}(x)\big)^p\Delta x}{\displaystyle\int_a^b h(x)g^q(x)\Delta x}.$$

This and $\dfrac{1}{p}+\dfrac{1}{q}=1$ imply

$$\int_a^b h(x)f(x)g(x)\Delta x \le \left(\int_a^b h(x)f^p(x)\Delta x\right)^{\frac{1}{p}}\left(\int_a^b h(x)g^q(x)\Delta x\right)^{\frac{1}{q}}.$$

Theorem 3.3. Let $h, f, g \in C_{rd}\big([a,b],[0,\infty)\big)$. Then

(a)
$$\left[\left(\int_a^b hf\Delta x\right)^r + \left(\int_a^b hg\Delta x\right)^r\right]^{\frac{1}{r}} \le \int_a^b h(f^r+g^r)^{\frac{1}{r}}\Delta x, \text{ if } r>1\,;$$

(b)

$$\left[\left(\int_a^b hf\Delta x\right)^r + \left(\int_a^b hg\Delta x\right)^r\right]^{\frac{1}{r}} \ge \int_a^b h(f^r+g^r)^{\frac{1}{r}}\Delta x, \text{ if } 0<r<1\,.$$

Proof. (a) Clearly, $\varphi(x) = (1 + x^r)^{\frac{1}{r}}$ is convex on $(0, \infty)$. Hence, by Theorem 2.2,

$$\left[1 + \left(\int_a^b h(x)f(x)\Delta x\right)^r\right]^{\frac{1}{r}} \leq \int_a^b h(x)(1 + f^r(x))^{\frac{1}{r}}\Delta x\,.$$

Letting h and f be replaced by $\dfrac{hf}{\int_a^b hf\Delta x}$ and $\dfrac{g}{f}$ in the above inequality, respectively, we get our desired result.

Similarly, we can prove case (b). ■

References

[1] R. Agarwal, M. Bohner, and A. Peterson. Inequalities on time scales: A survey, *Math. Inequal. Appl.*, 7:535–557, 2001.

[2] B. Aulbach and S. Hilger. Linear dynamic processes with inhomogeneous time scale in “Nonlinear Dynamics and Quantum Dynamical Systems (Gaussig, 1990)”, Volume 59 of Math. Res., pp. 9–20 Akademie-Verlag, Berlin, 1990.

[3] M. Bohner and A. Peterson. “Dynamic Equations on Time Scales” Birkhäuser, Boston, Basel, Berlin, 2001.

[4] O. Dunkel. Integral inequalities with applications to the calculus of variations, *Amer. Math.*, Monthly 31:326–337, 1924.

[5] G. Folland. “Real Analysis: Modern Techniques and Their Applications” John Wiley & Sons, Inc., New York, second edition, 1999.

[6] S. Hilger. Analysis on measure chains — A unified approach to continuous and discrete calculus, *Res. Math*, 18:28–56, 1990.

[7] B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaram. “Dynamic Systems on Measure Chains”, Kluwer Acadmic Publishers, Dordrecht, 1996.

[8] L. Maligranda. Why Hölder’s inequality should be called Rogers’ inequality, *Math. Inequal. Appl.*, 1:69–83, 1998.

[9] D.S. Mitrinovic. “Analytic Inequalities”, Springer-Verlag, New York/Heidelberg/ Berlin, 1970.